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Progress in Dynamo Theory: Kinematic and Dynamic Models

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Progress in dynamo theory: kinematic and dynamic models

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The results of dynamo theories that have been deduced during the last 10 years have often been used for deriving statements relating properties of the magnetic field with those of the core of the Earth. These results are critically analysed, and the reliability of such statements is discussed.

Introduction

Dynamo theory has been developed to answer the question of (i) the spatial structure and the time behaviour of the magnetic field of a cosmic body if the structure of that body is known, or, likewise, the inverse problem, (ii) the internal structure of a cosmic body if the magnetic field on the surface of that body is known.

It must be confessed, however, that we are still far from having achieved these aims, although there was clearly a breakthrough in the development of dynamo theory in the mid-1960s. This breakthrough was related to the question of the origin of the magnetic fields of cosmic bodies such as the Earth and the Sun, but was insufficiently productive with regard to the questions raised above.

At that time it was well known that if the magnetic field can be explained on the basis of the known laws of classical physics, it must be generated by a self-excited dynamo. For the explanation, however, the scientific world had to wait some decades, since J. Larmor in 1919 first proposed the self-excited dynamo mechanism, or one century, following the construction of the technical self-excited dynamo by W. v. Siemens in 1867.

The difference between the technical dynamo and a cosmic one like the Earth is obvious. Although simple with regard to the basic principle, the technical dynamo has a complicated topological structure. Isolated wires are wound in a quite definite way. A cosmic body such as the Earth, however, is a topologically simply connected body. For the topological simplicity of a cosmic dynamo, we have to pay with the complexity of the fields involved in the dynamo process, i.e. the magnetic field and the velocity field. Cowling's theorem from 1934, which states that dynamo excitation of axisymmetric magnetic fields is impossible, illuminates the situation.

As a most powerful method averaging procedures had been used in order to overcome the mathematical difficulties. I mention the successful attempts, which are known as the 'nearly symmetric dynamo' of Braginsky (1964) and 'mean field electrodynamics' by Steenbeck *et al.* (1966). For an illustration I sketch here the latter (for more details see Moffatt (1978) or Krause & Rädler (1980)).

Kinematic models

The governing equations are Maxwell's and the related constitutive equations:

$$\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t, \quad \nabla\times\boldsymbol{H} = \boldsymbol{j}, \quad \nabla\cdot\boldsymbol{B} = 0; \tag{1}$$

$$\boldsymbol{B} = \mu\boldsymbol{H}, \quad \boldsymbol{j} = \sigma(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}); \tag{2}$$

written down here in obvious notation. Bearing in mind that the fields involved are composed of a mean part, $\overline{\boldsymbol{B}}$, $\overline{\boldsymbol{u}}$, ..., and a fluctuating one, $\boldsymbol{B}'$, $\boldsymbol{u}'$, ..., we obtain by averaging the above equations

$$\nabla\times\overline{\boldsymbol{E}} = -\dot{\overline{\boldsymbol{B}}}, \quad \nabla\times\overline{\boldsymbol{H}} = \overline{\boldsymbol{j}}, \quad \nabla\cdot\overline{\boldsymbol{B}} = 0; \tag{3}$$

$$\overline{\boldsymbol{B}} = \mu\overline{\boldsymbol{H}}, \quad \overline{\boldsymbol{j}} = \sigma(\overline{\boldsymbol{E}}+\overline{\boldsymbol{u}}\times\overline{\boldsymbol{B}}+\mathscr{E}); \tag{4}$$

where

$$\mathscr{E} = \overline{\boldsymbol{u}'\times\boldsymbol{B}'}. \tag{5}$$

The difference of the equations for the mean fields from the original equations lies in the appearance of the additional electromotive force $\mathscr{E}$ in Ohm's law. The main problem of mean-field electrodynamics is the representation of $\mathscr{E}$ as a function of the mean quantities $\overline{\boldsymbol{u}}$, $\overline{\boldsymbol{B}}$, $\overline{\boldsymbol{u}'^2}$,... . Although this problem is rather difficult and not yet completely solved, I am able to present here for a spherical convective shell on a rotating body, which is of interest in our context, the expression

$$\mathscr{E} = -\alpha_1(\boldsymbol{g}\cdot\boldsymbol{\Omega})\cdot\overline{\boldsymbol{B}}-\alpha_2[(\boldsymbol{g}\cdot\overline{\boldsymbol{B}})\,\boldsymbol{\Omega}+(\boldsymbol{\Omega}\cdot\overline{\boldsymbol{B}})\,\boldsymbol{g}]-\gamma\boldsymbol{g}\times\overline{\boldsymbol{B}}-\gamma_1(\boldsymbol{g}\times\boldsymbol{\Omega})\times\overline{\boldsymbol{B}}$$
$$-\beta\nabla\times\overline{\boldsymbol{B}}-\beta_2\nabla(\boldsymbol{\Omega}\cdot\overline{\boldsymbol{B}})-\beta_3(\boldsymbol{\Omega}\cdot\nabla)\,\overline{\boldsymbol{B}}, \tag{6}$$

which is rather general in case linearity in the vectors $\overline{\boldsymbol{B}}$, $\boldsymbol{g}$ (denoting the radial stratification) and $\boldsymbol{\Omega}$ (angular velocity) is required. Equation (6) contains seven scalars, α_1, α_2, ..., β, which depend on the characteristics of the turbulent motion. For dynamo theory, however, it is mainly the first one that is responsible for the excitation that appears. It is the one that describes an electromotive force parallel to the magnetic field, a phenomenon that is indeed unknown in normal electrodynamics, which has become known as the α-effect. One should note in Ohm's law,

$$\boldsymbol{j} = \sigma(\boldsymbol{E}+\alpha\boldsymbol{B}), \tag{7}$$

α must be a pseudo-scalar, that means that we necessarily need stratification, and rotation in order to construct such a quantity:

$$\alpha = -\alpha_1(\boldsymbol{g}\cdot\boldsymbol{\Omega}). \tag{8}$$

It can be easily seen that self-excitation would appear in a medium obeying an Ohm's law (7) (cf. Krause & Rädler 1980, p. 12).

Two main types of dynamo have been discussed in connection with the explanation of the Earth's magnetic field: the α^2-dynamo and the $\alpha\omega$-dynamo; in the latter case the production of the toroidal field from the poloidal is assumed to be due to differential rotation. Independent of the model, which is preferred, convection and α-effect is needed and consequently we can draw some conclusions concerning the Earth's core.

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Character of the convective motions

Let us first state that the existence of the Earth's magnetic field gives clear evidence of internal motions and consequently of a fluid core. The behaviour of the magnetic field is governed by the induction equation

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + (1/\mu\sigma)\,\Delta \boldsymbol{B}. \tag{9}$$

In order to prevent decay the first term on the right has to overcome the second one. This allows for the estimate

$$R_{\mathrm{m}} = \mu\sigma u R > 1, \tag{10}$$

which means that for the Earth's core ($\mu = \mu_0$, $\sigma \approx 10^5\ \Omega^{-1}\ \mathrm{m}^{-1}$, $R \approx 3 \times 10^6$ m)

$$u > 3 \times 10^{-6}\ \mathrm{m\ s^{-1}}. \tag{11}$$

In this way we have a lower bound for the velocity inside the core.

In order to have an α-effect the convective motion has to fulfil further conditions, as follows. Let λ_{cor}, τ_{cor} denote the size and lifetime of convection cells. The convection must be significantly influenced by the rotational motion, i.e. the lifetime must be comparable with or larger than the rotational period, say $\tau_{\mathrm{cor}} \gtrsim \frac{1}{2}P_{\mathrm{rot}}$. Furthermore, the magnetic field must be significantly influenced by the convectional motion, i.e. $\mu\sigma\lambda^2_{\mathrm{cot}} \gtrsim \frac{1}{2}P_{\mathrm{rot}}$. For the Earth we conclude:

$$\tau_{\mathrm{cor}} \gtrsim 5.10^4\ \mathrm{s}; \quad \lambda_{\mathrm{cor}} \gtrsim 1\ \mathrm{km}. \tag{12}$$

In addition, dynamo excitation, as a typical instability, does not appear unless the inductive effects involved are sufficiently strong. Let $C_\alpha = \mu\sigma\alpha R$ and $C_\omega = \mu\sigma R^2 \Delta\omega$ ($\Delta\omega$ denotes a characteristic value of the differential rotation) denote parameters describing these effects. It is known that α^2-dynamos will work if

$$C_\alpha > C_{\alpha,\,\mathrm{crit}}, \tag{13}$$

and $\alpha\omega$-dynamos if

$$C_\alpha C_\omega > (C_\alpha C_\omega)_{\mathrm{crit}}, \tag{14}$$

where the numerical values of the critical parameters depend on details of the models considered.

The following conclusion can be drawn: it is generally believed that Earth and Venus are to a large extent of the same structure. Let us therefore assume that Venus has a core of about the same size and structure. The rotational periods, however, of both these objects differ by a factor of more than 200. Since the α-effect and differential rotation are due to the bulk rotation we may conclude that

$$(C_\alpha)_{\mathrm{Venus}} \approx \tfrac{1}{200}(C_\alpha)_{\mathrm{Earth}}; \quad (C_\alpha C_\omega)_{\mathrm{Venus}} \approx (\tfrac{1}{200})^2 (C_\alpha C_\omega)_{\mathrm{Earth}}. \tag{15}$$

These relations might indicate that Venus is subcritical with respect to dynamo instability, and therefore has no global magnetic field.

Energy source of the Earth's dynamo

Thermal convection, precession and compositional convection have been proposed in the past as candidates for the energy source of the Earth's dynamo. Closer examination casts some doubt on whether or not the first two will really work. The investigations of Loper & Roberts (1978; see also Frank 1982), especially, make it plausible that the source of power is the gravitational potential energy released by the irreversible redistribution of mass associated with the growth of the inner core. The power available is estimated to be

$$P \approx 4 \times 10^{11}\ \mathrm{W}. \tag{16}$$

I shall not discuss here which mechanism is the most plausible, but again underline that the existence of the magnetic field necessarily leads to the conclusion that inside the Earth's core is convection with cells of a size of at least 1 km, and probably larger as we shall soon see. Consequently, deductions that state the core to be stably stratified (Higgins & Kennedy 1971) must be rejected.

The estimate (16) shall now be used to impose a stronger condition. We obtain for energy dissipated in the form of joule heat

$$\int \frac{j^2}{\sigma}\,\mathrm{d}V \approx \frac{l}{\sigma}\left(\frac{B}{\mu L}\right)^2 R^3, \tag{17}$$

where L denotes a length scale of the magnetic field. We have $L \approx \lambda_{\text{cor}}$ for the poloidal field and for the toroidal field either $L \approx \lambda_{\text{cor}}$ or $L \approx R$ dependent on whether an α^2-dynamo or an $\alpha\omega$-dynamo is considered. Let q be defined by $R = qL$ and r by $B_{\text{tor}} = rB_{\text{pol}}$. From (17) we have

$$\int \frac{j^2}{\sigma}\,\mathrm{d}V \approx \frac{B^2_{\text{pol}}R}{\mu^2\sigma}\,(q^2_{\text{pol}}+r^2q^2_{\text{tor}}) \approx 3\times 10^7(q^2_{\text{pol}}+r^2q^2_{\text{tor}})\ \mathrm{W} < 4\times 10^{11}\ \mathrm{W}, \tag{18}$$

since we know from observations $B_{\text{pol}} \approx 1$ mT. For α^2-dynamos it is known that $B_{\text{pol}} \approx B_{\text{tor}}$. From (18) we find that $q_{\text{pol}} \approx q_{\text{tor}} < 10^{-2}$,

$$\lambda_{\text{cor}} > 10\ \text{km}, \tag{19}$$

an even larger size for the convection cells than stated in (12). For $\alpha\omega$-dynamos, we also find (19) and in addition $r < 10^{-2}$, i.e. the toroidal field cannot be larger than some tens of milliteslas.

In conclusion, it may be stated that to have a production of Joule heat within reasonable limits, the size of convection cells cannot be smaller than 10 km and the toroidal field cannot be larger than some tens of milliteslas.

Westward drift

Concerning the westward drift of the geomagnetic field I mention here that in connection with the $\alpha\omega$-dynamo an explanation was given in terms of Braginsky's MAC-waves. For the α^2-dynamo the excitation of a longitudinally migrating, non-axisymmetric **B**-mode in addition to the axisymmetric dipole-like field is plausible because the eigenvalues of these fields are close together. The simultaneous appearance of these field-modes could explain the westward drift as well as the inclination of the dipole axis with respect to the rotation axis (Roberts & Stix 1972).

Nonlinear dynamo theory

In attempting to answer the questions posed at the beginning of this paper we are confronted with the complete nonlinear problem that arises when, in addition to the electrodynamics, the dynamics of the system is considered. In the simplest case of an incompressible medium we have, in addition to the induction equation (9), the Navier–Stoke's equation

$$\rho(\partial \boldsymbol{u}/\partial t+(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}) = -\nabla P+\boldsymbol{j}\times\boldsymbol{B}+\boldsymbol{F}+\rho\nu\Delta\boldsymbol{u}, \quad \nabla\cdot\boldsymbol{u} = 0, \tag{20}$$

where, apart from obvious rotations, $\boldsymbol{F}$ denotes the driving force, which is mostly assumed to be buoyancy.

It should first be stated that there is little hope that simple and unique relations between the field strength of the excited field and the physical parameters of a cosmic body do really exist. To explain this pessimism I present the following result of kinematic theory.

Within the frame of this theory we can answer the question for the ***B***-mode, which is the most easily excited. For a slow rotation the α-tensor is of the form $\alpha_{ij} = \alpha_0 \delta_{ij}$ (pure α-effect) and the easily excited magnetic field mode is axisymmetric, steady and of dipole type (Steenbeck & Krause 1969). For a high rotational rate, however, the α-tensor takes the form $\alpha_{ij} = \alpha_0 (\delta_{ij} - \omega_i \omega_j / \omega^2)$ (Moffatt 1970; Rüdiger 1978; Busse & Miin 1979) and now a ***B***-mode is excited first, which is non-axisymmetric and longitudinally migrating (Rüdiger 1980). Hence it is clearly to be seen that there is no simple proportionality between the amplitude of the excited magnetic field and the rate of rotation. The field completely changes its geometrical structure and time behaviour when the rotational speed increases.

But in spite of this pessimism I should not fail to report attempts of predictions, which are derived from nonlinear models based on more or less speculative assumption.

Busse (1976) derived a relation of that kind from an investigation of the annulus model, which was remarkable in that the predicted amplitudes fitted the observed planetary magnetic fields surprisingly well. The theoretical support, however, for this approximate relation is weak (Busse 1979); in particular, a recent quasi-linear investigation of Cuong & Busse (1981) in which they outlined an attack on the fully nonlinear problem, revealing certain significant differences between the annulus and the spherical dynamo.

Another quasi-linear theory was developed by Soward (see Soward 1979) in terms of the convection-driven dynamo. On the basis of these investigations it was suggested that the Earth's dynamo operates at

$$B^2 \sigma / \rho \omega \approx 1, \tag{21}$$

which means a maximal field strength of about 10 mT. A rather strong toroidal field is predicted. The same is true for the 'model z' of Braginsky (1975, 1978), where for 'model z' a field strength up to nearly 0.1 T is looked upon as possible.

Another kind of investigation is based on ordinary nonlinear differential equations, derived from the basic equations by rigorous simplifications. To some extent a physical interpretation is possible in terms of Rikkitake's two-disc dynamo (Rikkitake 1958). In particular the investigations of Cook & Roberts (1971), and those made by Robbins (1976, 1977), have shown that the solutions show special features, which recently have become known as 'strange attractor' behaviour. It is interesting to note here that the solutions, as functions of time, show a quasi-random change of sign, which with regard to the geomagnetic field may be interpreted as randomly appearing reversals. Thus the consideration of these highly simplified models offer explanations of the reversals of the magnetic field of the Earth as an intrinsic property of the nonlinear dynamo.

Concluding remarks

Although, as I have tried to show here, dynamo theory as it stands today gives some insight into relations between the magnetic field and the structure of a cosmic body, a final and definite answer to questions of this kind is still required. As long as we are still some distance from having solved the complete nonlinear equations, we are in a situation about which we can say that the more nonlinearity is included, the more inaccurate are the deductions and, as a consequence, the more speculative are the predictions.

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